PATTERN RIGIDITY AND THE HILBERT-SMITH CONJECTURE

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Abstract. In this paper we initiate a study of the topological group $PPQI(G, H)$ of pattern-preserving quasi-isometries for G a hyperbolic Poincare duality group and H an infinite quasiconvex subgroup of infinite index in G. Suppose ∂G admits a visual metric d with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$.
a) If Q_u is a group of pattern-preserving uniform quasi-isometries (or more generally any locally compact group of pattern-preserving quasi-isometries) containing G, then G is of finite index in Q_u.
b) If instead, H is a codimension one filling subgroup, and Q is any group of pattern-preserving quasi-isometries containing G, then G is of finite index in Q. Moreover, **(Topological Pattern Rigidity)** if L is the limit set of H, $\mathcal{L}$ is the collection of translates of L under G, and Q is any pattern-preserving group of *homeomorphisms* of ∂G preserving $\mathcal{L}$ and containing G, then the index of G in Q is finite.
We find analogous results in the realm of relative hyperbolicity, regarding an equivariant collection of horoballs as a symmetric pattern in a *hyperbolic* (not relatively hyperbolic) space. Combining our main result with a theorem of Mosher-Sageev-Whyte, we obtain QI rigidity results.

An important ingredient of the proof is a version of the Hilbert-Smith conjecture for certain metric measure spaces, which uses the full strength of Yang's theorem on actions of the p-adic integers on homology manifolds. This might be of independent interest.

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Contents

1. Preliminaries

1.1. **Statement of Results.** In this paper we start studying the full group of 'pattern-preserving quasi-isometries' for pairs (G, H) where G is a (Gromov) hyperbolic group and H an infinite quasiconvex subgroup of infinite index in G. In [Gro93] Gromov proposed the project of classifying finitely generated groups up to quasi-isometry, as well as the study of the group $QI(X)$ of quasi-isometries of a space X, where two quasi-isometries are identified if they lie at a bounded distance from each other. A class of groups where any two members are quasi-isometric if and only if they are commensurable is said to be quasi-isometrically rigid. However, any class of groups acting freely, cocompactly and properly discontinuously on some fixed proper hyperbolic metric space $\mathbf{H}$ are quasi-isometric to $\mathbf{H}$ and hence to each other. In this context (or in a context where quasi-isometric rigidity is not known) it makes sense to ask a relative version of Gromov's question. To obtain rigidity results, we impose additional restrictions on the quasi-isometries by requiring that they preserve some additional structure given by a 'symmetric pattern' of subsets. A 'symmetric pattern' of subsets roughly means a G-equivariant collection $\mathcal{J}$ of convex (or uniformly quasiconvex) cocompact subsets in $\mathbf{H}$. Then the relative version of Gromov's question for classes of pairs (G, H) was formulated by Mosher-Sageev-Whyte [MSW04] as the following *pattern rigidity* question:

Question 1.1. *Given a quasi-isometry q of two such pairs (G_i, H_i) $(i = 1, 2)$ pairing a (G_1, H_1)-symmetric pattern $\mathcal{J}_1$ with a (G_2, H_2)-symmetric pattern $\mathcal{J}_2$, does there exist an abstract commensurator I which performs the same pairing?*

The study of this question was initiated by Schwartz [Sch95], [Sch97], where G is a lattice in a rank one symmetric space. The paper [Sch95] deals with symmetric patterns of convex sets whose limit sets are single points, and [Sch97] deals with symmetric patterns of convex sets (geodesics) whose limit sets consist of two points. In [BM08], Biswas and Mj generalized Schwarz' result to certain Duality and PD subgroups of rank one symmetric spaces. In [Bis09], Biswas completely solved the pattern rigidity problem for G a uniform lattice in real hyperbolic space and H any infinite quasiconvex subgroup of infinite index in G. However, all these papers used, in an essential way, the linear structure of the groups involved, and the techniques fail for G the fundamental group of a general closed negatively curved manifold. (This point is specifically mentioned by Schwartz in [Sch97]). Further, the study in [Sch95], [Sch97], [BM08], [Bis09] boils down to the study of a single pattern-preserving quasi-isometry between pairs (G_1, H_1) and (G_2, H_2). We propose a different perspective in this paper by studying the full group $PPQI(G, H)$ of pattern-preserving (self) quasi-isometries of a pair (G, H) for G a hyperbolic group and H any infinite quasiconvex subgroup of infinite index. The features of G that

we shall use are general enough to go beyond the linear context while at the same time strong enough to ensure rigidity in certain contexts. Some of the ingredients of this paper are:
1) The boundary of a Poincare duality hyperbolic group is a generalized manifold by a Theorem of Bestvina-Mess [BM91].
2) The algebraic topology of homology manifolds imposes restrictions on what kinds of groups may act on them by Theorems of Newman [New31], Smith [Smi41] and Yang [Yan60].
3) Boundaries of hyperbolic groups equipped with the visual metric also have a metric measure space structure with the property that they are Ahlfors regular.
4) Quasiconformal analysis can be conducted in the general context of Ahlfors regular metric spaces.
5) A combinatorial cross-ratio can be constructed on the boundary of a hyperbolic group in the presence of a codimension one subgroup.
Of these ingredients, the first two come from (a somewhat forgotten chapter of) algebraic topology, the next two from a very active new area of analysis on metric measure spaces, while the last comes from geometric group theory proper. Topological actions of finite groups on manifolds and homological consequences of actions of p-adics on manifolds form the two main ingredients for a proof of the Hilbert-Smith conjecture for bi-Lipschitz [RS97] and quasiconformal [Mar99] actions. We first generalize the result of Martin [Mar99] to Ahlfors regular metric spaces and obtain the following.

Theorem 2.6 : Let (X, d, μ) be an Ahlfors regular metric measure space such that X is a homology manifold and $dim_H(X) < dim_t(X) + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension. Then (X, d, μ) does not admit an effective $Z_{(p)}$-action by quasiconformal maps, where $Z_{(p)}$ denotes the p-adic integers. Hence any finite dimensional locally compact group acting effectively on (X, d, μ) by quasiconformal maps is a Lie group.

The last statement follows from the first by standard arguments [Mar99]. As in [Mar99], there is no assumption on the uniformity of the quasiconformal maps in $Z_{(p)}$. The analogue of Theorem 2.6 is *false* for purely topological actions [RW56] on homology manifolds. Hence the quasiconformality assumption is crucial here. The statement that a topological manifold does not admit an effective topological $Z_{(p)}$ action is the famous Hilbert-Smith conjecture (but does not imply Theorem 2.6).

Theorem 2.6 will be a crucial ingredient in our approach to pattern rigidity. Another property we shall be investigating in some detail is the notion of 'topological infinite divisibility'. The notion we introduce is somewhat weaker than related existing notions in the literature. In this generality, we prove

Propositions 3.7 and 3.9 Let G be a hyperbolic group and H an infinite quasiconvex subgroup of infinite index in G. Then
a) any group of pattern-preserving quasi-isometries is totally disconnected and contains no topologically infinitely divisible elements.
b) If G is a Poincare duality group, the group $QI(G)$ of quasi-isometries cannot contain arbitrarily small torsion elements.

We obtain stronger results under the assumption that G is a Poincare duality group (e.g. the fundamental group of a closed negatively curved manifold) with mild restrictions on the visual metric on its boundary.
Theorems 3.11, 4.7 and 4.9 Let G be a hyperbolic Poincare duality group and

H an infinite quasiconvex subgroup of infinite index in G. Suppose further that for some visual metric on ∂G, $dim_H(\partial G) < dim_t(\partial G) + 2$, where dim_H and dim_t denote Hausdorff and topological dimension respectively.
a) If Q_u is a group of pattern-preserving uniform quasi-isometries (or more generally any locally compact group of pattern-preserving quasi-isometries) containing G, then G is of finite index in Q_u.
b) If further, H is a codimension one filling subgroup, and Q is any group of (not necessarily uniform) pattern-preserving quasi-isometries containing G, then G is of finite index in Q.
c) **Topological Pattern Rigidity** Under the assumptions of (b), let L be the limit set of H and $\mathcal{L}$ be the collection of translates of L under G. Let Q be any pattern-preserving group of homeomorphisms of ∂G preserving $\mathcal{L}$ and containing G. Then the index of G in Q is finite.

Theorem 4.9 is a generalization of a Theorem of Casson-Bleiler [CB88] and Kapovich-Kleiner [KK00] to all dimensions. Casson-Bleiler [CB88] and Kapovich-Kleiner [KK00] proved Theorem 4.9 for G the fundamental group of a surface and H an infinite cyclic subgroup corresponding to a filling curve.

We also derive QI rigidity results for relatively hyperbolic groups (e.g fundamental groups of non-compact negatively curved manifolds of finite volume), by deriving analogues of Theorem 3.11 for symmetric patterns of horoballs and combining it with a Theorem of Behrstock-Drutu-Mosher [BDM09]. The hypotheses in the following Theorem are satisfied by fundamental groups of finite volume complete non-compact manifolds of pinched negative curvature and dimension bigger than 2.

Theorems 5.6 Let G be a finitely generated group (strongly) hyperbolic relative to a finite collection $\mathcal{H}$ of finitely generated subgroups H such that
1) each $H \in \mathcal{H}$ is not (strongly) relatively hyperbolic
2) $\partial(G, \mathcal{H})$ is a cohomology manifold.
3) $dim_H(\partial(G, \mathcal{H})) < dim_t(\partial(G, \mathcal{H})) + 2$, where dim_H and dim_t denote Hausdorff and topological dimension respectively.
Let Γ be a Cayley graph of G with respect to a finite generating set. Let Q be a group of uniform quasi-isometries of Γ containing G. Then G is of finite index in Q. In particular, $Q \subset Comm(G)$, where $Comm(G)$ denotes the abstract commensurator of G.

The author learnt the following Scholium from Professor M. Gromov [Gro09].

Scholium 1.2. *If two discrete groups can be embedded in the same locally compact group nicely, they are as good as commensurable.*

A partial aim of this paper is to make Scholium 1.2 precise in the context of pattern rigidity. It follows from Theorems 3.11 and 5.6 that in the context of pattern rigidity or QI rigidity of (fundamental groups of) finite volume complete non-compact manifolds of pinched negative curvature and dimension bigger than 2, ‘as good as’ can be replaced by ‘actually’ in Scholium 1.2. Thus, Theorems 3.11 and 5.6 reduce the problem of pattern rigidity to the weaker problem of embedding two groups simultaneously in the same locally compact group and Theorem 4.7 carries out this embedding under certain hypotheses.

1.2. **Dotted geodesic metric spaces and Patterns.**

Definition 1.3. *A dotted metric space is a metric space X, where $d(x,y)$ is an integer for all $x,y \in X$. A dotted geodesic metric space is a dotted metric space X, such that for all $x,y \in X$, there exists an isometric map $\sigma\colon [0,d(x,y)] \cap \mathbb{Z} \to X$ with $\sigma(0) = x$ and $\sigma(d(x,y)) = y$. A dotted metric space is proper if every ball $N_k(x)$ is finite.*

The following easy observation will turn out to be quite useful.

Lemma 1.4. *Let G be a closed set of (K,ϵ) quasi-isometries of a proper dotted metric space X such that there exists $x \in X$ and $C \geq 0$ such that $d(x,g(x)) \leq C$. Then G is compact in the compact open topology. Hence any group of uniform quasi-isometries of a proper dotted metric space X is locally compact.*

Proof: Since $N_C(x)$ is finite, it suffices to prove that $\{g \in G\colon g(x) = y\}$ is compact. The Lemma now follows by passing to convergent subsequences on a sequence of larger and larger balls, and diagonalizing, as all balls are finite. $\Box$

Lemma 1.4 may be thought of as a coarsening of the fact that the stabilizer of a point in the isometry group of a Riemannian manifold is compact.

Definition 1.5. *A* **symmetric pattern** *of closed convex (or quasiconvex) sets in a hyperbolic metric space* $\mathbf{H}$ *is a G-invariant countable collection $\mathcal{J}$ of quasiconvex sets such that*
1) G acts cocompactly on $\mathbf{H}$*.*
2) The stabilizer H of $J \in \mathcal{J}$ acts cocompactly on J.
3) $\mathcal{J}$ is the orbit of some (any) $J \in \mathcal{J}$ under G.

This definition is slightly more restrictive than Schwartz' notion of a symmetric pattern of geodesics, in the sense that he takes $\mathcal{J}$ to be a finite union of orbits of geodesics, whereas Condition (3) above forces $\mathcal{J}$ to consist of one orbit. All our results go through with the more general definition, where $\mathcal{J}$ is a finite union of orbits of closed convex (or quasiconvex) sets, but we restrict ourselves to one orbit for expository ease.

Suppose that $(X_1,d_1),(X_2,d_2)$ are metric spaces. Let $\mathcal{J}_1,\mathcal{J}_2$ be collections of closed subsets of X_1,X_2 respectively. Then d_i induces a *pseudo-metric* (which, by abuse of notation, we continue to refer to as d_i) on $\mathcal{J}_i$ for $i = 1,2$. This is just the ordinary (not Hausdorff) distance between closed subsets of a metric space.

In particular, consider two hyperbolic groups G_1,G_2 with quasiconvex subgroups H_1,H_2, Cayley graphs Γ_1,Γ_2. Let $\mathcal{L}_j$ for $j = 1,2$ denote the collection of translates of limit sets of H_1,H_2 in $\partial G_1,\partial G_2$ respectively. Individual members of the collection $\mathcal{L}_j$ will be denoted as L_i^j. Let $\mathcal{J}_j$ denote the collection $\{J_i^j = J(L_i^j) : L_i^j \in \mathcal{L}_j\}$ of joins of limit sets. Recall that the join of a limit set Λ_i is the union of bi-infinite geodesics in Γ_i with end-points in Λ_i. This is a uniformly quasiconvex set and lies at a bounded Hausdorff distance from the Cayley graph of the subgroup H_i. Following Schwartz [Sch97], we define:

Definition 1.6. *A bijective map ϕ from $\mathcal{J}_1 \to \mathcal{J}_2$ is said to be uniformly proper if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that*
1) $d_{G_1}(J(L_i^1),J(L_j^1)) \leq n \Rightarrow d_{G_2}(\phi(J(L_i^1)),\phi(J(L_j^1))) \leq f(n)$
2) $d_{G_2}(\phi(J(L_i^1)),\phi(J(L_j^1))) \leq n \Rightarrow d_{G_1}(J(L_i^1),J(L_j^1)) \leq f(n)$.

When $\mathcal{J}_i$ consists of all singleton subsets of Γ_1,Γ_2, we shall refer to ϕ as a uniformly proper map from Γ_1 to Γ_2.

The following Theorem was proven in [Mj08] (See Lemmas 3.3, 3.4 and Theorem 3.5 of [Mj08]).

Theorem 1.7. [Mj08] *Let H be an infinite quasiconvex subgroup of a hyperbolic group G such that H has infinite index. Let Γ be a Cayley graph of G with metric d. Let L be the limit set of H and $\mathcal{L}$ be the collection of translates of L under G. There exists a finite collection $L_1, \cdots L_n$ of elements of $\mathcal{L}$ such that the following holds.*
For any K, ϵ, there exists a C such that if $\phi : \Gamma \to \Gamma$ be a pattern-preserving (K, ϵ)-quasi-isometry of Γ with $\partial\phi(L_i) = L_i$ for $i = 1 \cdots n$, then $d(\phi(1), 1) \leq C$.

Definition 1.8. *The group $PP(G, H)$ of pattern-preserving maps for a pair (G, H) as above is defined as the group of homeomorphisms of ∂G that preserve the collection of translates $\mathcal{L}$. The group $PPQI(G, H)$ of pattern-preserving quasi-isometries for a pair (G, H) as above is defined as the subgroup of $PP(G, H)$ consisting of homeomorphisms h of ∂G such that $h = \partial\phi$ for some quasi-isometry $\phi : \Gamma \to \Gamma$.*

Proposition 1.9. [Mj08] *The collection $\mathcal{L}$ is discrete in the Hausdorff topology on the space of closed subsets of ∂G, i.e. for all $L \in \mathcal{L}$, there exists $\epsilon > 0$ such that $N_\epsilon(L) \cap \mathcal{L} = L$, where $N_\epsilon(L)$ denotes an ϵ neighborhood of L in the Hausdorff metric.*

Combining Lemma 1.4 with Theorem 1.7, we get

Corollary 1.10. *Let H be an infinite quasiconvex subgroup of a hyperbolic group G such that H has infinite index. Let Γ be a Cayley graph of G with metric d and Q be any group of uniform quasi-isometries of (the vertex set of) Γ. Let L be the limit set of H and $\mathcal{L}$ be the collection of translates of L under G. There exists a finite collection $L_1, \cdots L_n$ of elements of $\mathcal{L}$ such that $Q_0 = \cap_{i=1\cdots n} Stab(L_i)$ is compact, where $Stab(L_i)$ denotes the stabilizer of L_i in Q.*

Also, since the group of pattern preserving homeomorphisms of ∂G is a closed subgroup, as is the set of boundary values of any collection of uniform quasi-isometries, we have

Corollary 1.11. *Let H be an infinite quasiconvex subgroup of a hyperbolic group G such that H has infinite index. Let Γ be a Cayley graph of G with metric d and Q be any group of uniform pattern-preserving quasi-isometries of (the vertex set of) Γ. Then Q is locally compact.*

Proof: We include a slightly different direct proof here. The collection $\mathcal{L}$ is discrete by Proposition 1.9. Consider the finite collection $L_1, \cdots L_n$ in Corollary 1.10. There exists $\epsilon > 0$ such that $N_\epsilon(L_i) \cap \mathcal{L} = L_i$ for all $i = 1 \cdots n$. Define

$$N_\epsilon(Id) = \{q \in Q \colon d_{\partial G}(x, \partial\phi(x)) \leq \epsilon \text{ for all } x \in \partial G\}$$

where $d_{\partial G}$ denotes some visual metric on ∂G. Then $N_\epsilon(Id) \subset Q_0 = \cap_{i=1\cdots n} Stab(L_i)$, which is compact. Hence the Corollary. $\Box$

1.3. **Boundaries of Hyperbolic metric spaces and the Newman-Smith Theorem.** Let L be one of the rings $\mathbb{Z}$ or $\mathbb{Z}_p$, for p a prime.

Definition 1.12. *(*[Bre97]*, p.329) An m-*dimensional homology manifold over L *(denoted m-*hm_L*) is a locally compact Hausdorff space X with finite homological dimension over L, that has the local homology properties of a manifold, i.e. for all*

$x \in X$, $H_n(X, X \setminus \{x\}) = L$ and $H_i(X, X \setminus \{x\}) = 0$ for $i \neq n$.
Further, if X is an m-hm$_L$ and $H^c_*(X; L) \cong H^c_*(\mathbb{S}^m; L)$ then X is called a generalized m-sphere *over L.*

For homology manifolds, the existence of a local orientation was proven by Bredon in [Bre60].

The related notion of a *cohomology manifold over L*, denoted m-cm$_L$ is defined by Bredon in [Bre97], p. 373. If $L = \mathbb{Z}_p$, a connected space X is an n-cm$_L$ if and only it if it is an n-hm$_L$ and is locally connected ([Bre97], p. 375 Theorem 16.8, footnote).

We shall be using the following Theorem which is a result that follows from work of Bestvina-Mess [BM91] and Bestvina [Bes96] (See also Swenson [Swe99] , Bowditch [Bow98a] and Swarup [Swa96]).

Theorem 1.13. *Boundaries ∂G of PD(n) hyperbolic groups G are locally connected homological manifolds (over the integers) with the homology of a sphere of dimension $(n-1)$. Further, if G acts freely, properly discontinuously, cocompactly on a contractible complex X then $H^{LF}_n(X) = H_{n-1}\partial G$, where H^{LF}_n denotes locally finite homology.*

We shall be using Theorem 1.13 in conjunction with the following Theorem of Newman and Smith, which as stated below is a consequence of the work of several people (see below).

Theorem 1.14. *(Newman* [New31]*, Smith* [Smi41]*) Let (X, d) be a compact Z_p-cohomology manifold for all p, having finite topological (covering) dimension, and equipped with a metric d. There exists $\epsilon > 0$ such that if Z_p acts effectively on X (for some p), then the diameter of some orbit is greater than ϵ.*

Newman proved the above Theorem for closed orientable manifolds [New31]. Smith [Smi41] generalized it to locally compact spaces satisfying certain homological regularity properties. Building on work of Yang [Yan58], Conner and Floyd ([CF59], Corollary 6.2) proved that cohomological manifolds of finite topological (covering) dimension satisfy the regularity properties required by Smith's theorem. (The theorem also holds for a somewhat more general class of spaces, called 'finitistic spaces' by Bredon in [Bre72], but we shall not require this). For a historical account of the development of the theory of generalized manifolds and their connection with Smith manifolds see [Ray78].

We shall also be using a theorem on homological consequences of actions of p-adic transformation groups on homology manifolds.

Theorem 1.15. *(Yang* [Yan60]*) Let X be a compact homology n-manifold admitting an effective K-action, where $K = Z_{(p)}$ is the group of p-adic integers. Then the homological dimension of X/K is $n+2$.*

2. Ahlfors Regular Metric Measure Spaces

Let (X, d, μ) be a compact metric measure space, i.e. a metric space equipped with a Borel measure. We say that X is Ahlfors Q-regular, if there exists $C \geq 1$ such that for all $0 \leq r \leq dia(X)$, and any ball $B_r(x) \subset X$, the measure $\mu(B_r(x))$ satisfies $\frac{1}{C} r^Q \leq \mu(B_r(x)) \leq C r^Q$. When Q is omitted we assume that Q is the Hausdorff dimension and μ the Hausdorff measure. The relevance to the present paper comes from the following.

Theorem 2.1. *(***Coornaert** [Coo93]*) Let G be a hyperbolic group. Then $(\partial G, d)$ is Ahlfors regular for any visual metric d.*

Definition 2.2. *Let $f : X \to Y$ be a homeomorphism between metric spaces (X, d_X) and (Y, d_Y). Then f is quasiconformal if there exists a constant $C \geq 1$ such that*

$$H_f(x) = limsup_{r \to 0^+} \frac{sup\{d_Y(f(x_1), f(x)) : d_X(x_1, x) \leq r\}}{inf\{d_Y(f(x_1), f(x)) : d_X(x_1, x) \geq r\}} \leq C$$

for all $x \in X$.

It is a standard fact that the boundary values of quasi-isometries of hyperbolic metric spaces are quasiconformal maps for any visual metric on the boundary.

We start with a rather general result about compact group actions on compact metric measure spaces.

Lemma 2.3. *Let (X, d, μ) be an Ahlfors regular compact metric measure space with Hausdorff dimension $\mathcal{Q} \geq 1$. Let K be a compact topological group acting by uniformly C-quasiconformal maps on (X, d, μ) and equipped with a Haar measure of unit mass. Let d_K be the average metric on X given by $d_K(x, y) = \int_K d(g(x), g(y))dg$. Then the Hausdorff dimension of (X, d_K, μ) does not exceed Q.*

Proof: We adapt an argument of Repovs-Scepin [RS97] and Martin [Mar99] to the present context. Assume after normalization, that the total Hausdorff measure of X is one.

Cover X by a family $\mathcal{B}_r$ of balls of radius $r > 0$ measured with respect to d. Refine this cover using the Besicovitch covering theorem (see [Maz56] for instance) to ensure bounded overlap, i.e. there exists M (independent of r) such that each $B \in \mathcal{B}_r$ intersects at most M elements of $\mathcal{B}_r$ nontrivially. As in [Mar99], it is enough to find a uniform bound (independent of r) for the sum $\Sigma_{B \in \mathcal{B}_r} (dia_{d_K} B)^{\mathcal{Q}}$ where $dia_{d_K} B$ is the diameter of B in the invariant metric d_K.

For each $B \in \mathcal{B}_r$ let c_B denote its center and z_B a point on the boundary of B with $dia_{d_K} B \leq 2d_K(c_B, z_B)$. Then

$\Sigma_{B \in \mathcal{B}_r} (dia_{d_K} B)^{\mathcal{Q}}$

$\leq 2^{\mathcal{Q}} \Sigma_{B \in \mathcal{B}_r} (\int_K d(g(c_B), g(z_B))dg)^{\mathcal{Q}}$

$\leq 2^{\mathcal{Q}} \Sigma_{B \in \mathcal{B}_r} \int_K d(g(c_B), g(z_B))^{\mathcal{Q}} dg$ (by Holder's inequality since $\mathcal{Q} \geq 1$ and the normalization condition that the total Hausdorff measure of X is one)

$\leq (2C)^{\mathcal{Q}} \Sigma_{B \in \mathcal{B}_r} \int_K inf_{y_B \in \partial B} d(g(c_B), g(y_B))^{\mathcal{Q}} dg$ (since K acts by uniformly C-quasiconformal maps)

$\leq (2C)^{\mathcal{Q}} \Sigma_{B \in \mathcal{B}_r} \int_K \mu(g(B))dg$ (since X is Ahlfors $\mathcal{Q}$-regular)

$= (2C)^{\mathcal{Q}} \int_K \Sigma_{B \in \mathcal{B}_r} \mu(g(B))dg$

$\leq M(2C)^{\mathcal{Q}} \int_K \mu(g(X))dg$ (where M is the number obtained from the Besicovitch covering theorem that balls have bounded overlap)

$= M(2C)^{\mathcal{Q}} \int_K \mu(X)dg$ (since g is a homeomorphism from X onto itself)

$= M(2C)^{\mathcal{Q}}$ (since K is equipped with a Haar measure of unit mass and by the normalization condition that the total Hausdorff measure of X is one)

This establishes a uniform bound (independent of r) for the sum $\Sigma_{B \in \mathcal{B}_r} (dia_{d_K} B)^{\mathcal{Q}}$ and completes the proof. $\square$

Theorem 2.4. *(See* [HR63] *, Ch II, Thm 7.1, Thm. 7.3 p. 60,).*
Let G be a topological group. Then there exists an exact sequence

$$1 \to G_0 \to G \to H \to 1$$

with G_0 the connected component of the identity in G and H totally disconnected. If G, and hence H, is locally compact, then H contains arbitrarily small compact open subgroups, i.e. for every neghborhood U of the identity in H, there is a compact open subgroup $K \subset U \subset H$.

Moreover the structure of G_0 is well-known thanks to Montgomery amd Zippin ([MZ55], Thm 4.6, which implies the famous theorem that a locally compact topological group without *small subgroups* is a Lie group.)

Theorem 2.5. *(*[MZ55]*, Thm 4.6) Let G_0 be the connected component of the identity in a locally compact topological group G. For each neighborhood U of the identity in G_0, there exists a compact normal subgroup $K \subset U$ such that the quotient group G_0/K is a Lie group.*

We are now in a position to prove the following.

Theorem 2.6. *Let (X, d, μ) be an Ahlfors regular metric measure space such that X is a homology manifold and $dim_H(X) < dim_h(X)+2$, where dim_H is the Hausdorff dimension and dim_h is the homological dimension. Then (X, d, μ) does not admit an effective $Z_{(p)}$-action by quasiconformal maps, where $Z_{(p)}$ denotes the p-adic integers. Hence any finite dimensional locally compact group acting effectively on (X, d, μ) by quasiconformal maps is a Lie group.*

Proof: Suppose not. Let $K = Z_{(p)}$ be the compact group of p-adic integers acting effectively on X by quasiconformal maps. For $g \in K$, let $C(g)$ be its quasiconformal constant. Let $U_i = \{g \in K, C(g) \leq i\}$. Then $K = \bigcup_i U_i$ is the union of a countable family of closed sets whose union is G. By the Baire category theorem there is some U_C with nonempty interior. Translating by an element h of K in U_C, we may assume that U_c (for some c depending on C and the quasiconformal constant of h) contains the identity. But any neighbourhood of the identity in K contains an isomorphic copy of K. Hence we have an action of K (replacing the original group by the isomorphic copy contained in the above neighborhood of the identity) on (X, d, μ) by *uniformly* c-quasiconformal maps. Henceforth K will denote this new copy of K acting by *uniformly* c-quasiconformal maps.

Let d_K be the average metric on X given by $d_K(x, y) = \int_K d(g(x), g(y))dg$. Then the Hausdorff dimension of (X, d_K, μ) does not exceed Q by Lemma 2.3. Then K acts on (X, d_K) by isometries. Hence the the orbit space X/K admits the well-defined metric $\rho([x], [y]) = d_K(K(x), K(y))$, where $[x], [y]$ denote the images of x, y under the quotient map by K. Let $P : X \to X/K$ be the natural quotient map. Since P is clearly 1-Lipschits, it cannot increase Hausdorff dimension. Hence the Hausdorff dimension of X/K is at most equal to $dim_H(X)$, the Hausdorff dimension of X, which in turn is less than $dim_h(X)+2$. Since topological dimension is majorized by Hausdorff dimension and homological dimension is majorized by topological dimension, it follows that the homological dimension of X/K is less than $dim_h(X) + 2$. This directly contradicts Yang's Theorem 1.15 which asserts that the homological dimension of X/K is equal to $dim_h(X) + 2$ and establishes the theorem.

The last statement follows from the first by standard arguments (see [RS97] or [Mar99] for instance). We outline the argument for the sake of completeness. Let G be any finite dimensional locally compact group acting effectively on X. By Theorem 2.5, the connected component of the identity G_0 of G is a Lie group. Also if K is a small normal compact subgroup then we may assume that K is totally disconnected (since G_0 has no small subgroups). If K is infinite, it must contain a copy of the p-adics (cf Theorem 2.4) or have arbitrarily small torsion elements. But $QI(X)$ cannot contain such subgroups or elements. $\Box$

Since quasi-isometries of a hyperbolic group G act by quasiconformal maps on the boundary $(\partial G, d)$, where d is a visual metric, we have the following by combining Theorems 1.13, 2.1 with Theorem 2.6.

Corollary 2.7. *Let G be a Poincare duality hyperbolic group and Q be a group of (boundary values of) quasi-isometries of G. Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Then Q cannot contain a copy of $Z_{(p)}$, where $Z_{(p)}$ denotes the p-adic integers. Hence if Q is finite dimensional locally compact, it must be a Lie group.*

Note: The assumption $1 \leq dim_H$ is superfluous here as the only PD group that violates this hypothesis is virtually cyclic, when the Corollary is trivially true.
The hypothesis $dim_H < dim_t + 2$ is clearly true for (lattices in) real hyperbolic space, where $dim_H = dim_t$ as well as complex hyperbolic space, where $dim_H = dim_t + 1$. Amongst rank one symmetric spaces these are the only ones of interest in the context of pattern rigidity as quaternionic hyperbolic space and the Cayley plane are quasi-isometrically rigid in light of Pansu's fundamental result [Pan89]. The Gromov-Thurston [GT87] examples too seem to provide cases where Hausdorff dimension is close to topological dimension.

3. Pattern Preserving Groups as Topological Groups

3.1. **Infinite Divisibility.** We begin with some easy classical facts about topological groups. A property we shall be investigating in some detail is the notion of 'topological infinite divisibility'. The notion we introduce is weaker than related existing notions in the literature.

Definition 3.1. *An element g in a topological group G will be called* **topologically infinitely divisible**, *if there exists a sequence of symmetric neighborhoods U of the identity, such that $\bigcap U = \{1\}$ and $g \in \bigcup_1^\infty U^n \subset G$ for all U. Similarly, a subgroup H of G is said to be* **topologically infinitely divisible**, *if there exists a sequence of symmetric neighborhoods U of the identity, such that $\bigcap U = \{1\}$ and $H \subset \bigcup_1^\infty U^n \subset G$ for all U.*

Definition 3.2. *A topological group is said to have* **arbitrarily small torsion elements** *if for every neighborhood U of the identity, there exists an element $g_i \in U$ and a positive integer n_i such that $g_i^{n_i} = 1$, and furthermore $g^m \in U$ for all $m \in \mathbb{N}$.*

Definition 3.3. *A topological group containing a cyclic dense subgroup is said to be* **monothetic**.

It is easy to see that a monothetic group is abelian (using continuity of multiplication). The following is a consequence of a structure theorem for 0-dimensional compact monothetic groups (See [HR63], Thm. 25.16, p. 408.)

Theorem 3.4. *Any infinite* 0*-dimensional (i.e. totally disconnected), compact, monothetic group* K *contains a copy of the* $\mathbf{a}$*-adic integers* $A_{\mathbf{a}}$*, where* $\mathbf{a} = \{a_1, a_2, \cdots\}$ *is a sequence of integers* $a_i > 1$*. Hence* K *must contain arbitrarily small torsion elements or a copy of the group* $Z_{(p)}$ *of* p*-adic integers.*

For the rest of this subsection Q will denote a group of pattern-preserving quasi-isometries.

Proposition 3.5. Q *has no non-trivial topologically infinitely divisible elements. More generally,* Q *does not contain any non-trivial infinitely divisible subgroups.*

Proof: Suppose $q \in Q$ is infinitely divisible. Then for every neighborhood U of 1 there exists $m \in \mathbb{N}$ such that $q \in U^m$. Now for any finite collection $L_1, \cdots, L_n \in \mathcal{L}$, there exists a neighborhood U of $1 \in Q$ such that if $g \in U$ then $g(L_i) = L_i$ for $i = 1, \cdots, n$. Hence for any finite collection $L_1, \cdots, L_n \in \mathcal{L}$, there exists neighborhood U of 1 and $m \in \mathbb{N}$ such that $q \in U^m$ and $g(L_j) = L_j$ for $g \in U, j = 1, \cdots, n$. We say $U(L_j) = L_j$ for $g \in U, j = 1, \cdots, n$. Therefore $q(L_j) = U^m(L_j) = L_j$ for all $L_j \in \mathcal{L}$. That is q stabilizes every $L \in \mathcal{L}$. If $x \in \partial G$, there exist $L_m \in \mathcal{L}$ such that $L_m \to \{x\}$ (the singleton set containing x) in the Hausdorff topology on ∂G. Therefore $q(\{x\}) = \{x\}$ for all $x \in \partial G$, i.e. q is the trivial element of Q. The same argument shows that Q has no non-trivial topologically infinitely divisible subgroups. $\Box$

Remark 3.6. *We state the second conclusion of Proposition 3.5 slightly differently. Let* U_i *be a decreasing sequence of symmetric neighborhoods of the identity in* Q *such that* $\bigcap_i U_i = \{1\}$*. Let* $\langle U_i \rangle = \bigcup_n U_i^n$*. Then* $\bigcap_i \langle U_i \rangle = \{1\}$*.*

Since a connected topological group is generated by any neighborhood of the identity, we obtain

Proposition 3.7. Q *is totally disconnected.*

Note that in Proposition 3.5 and Proposition 3.7 we **do not** need to assume that Q is a group of **uniform** quasi-isometries.

Remark 3.8. *Let* K *be a compact group of pattern-preserving quasi-isometries. Then a reasonably explicit structure of* K *may be given as a permutation group. Since* K *is compact it acts on the discrete set* $\mathcal{L}$ *with compact and hence finite orbits. Let* $\mathcal{L}_1, \mathcal{L}_2, \cdots$ *be a decomposition of* $\mathcal{L}$ *into disjoint orbits under* K*. Then* $K \subset \Pi_i S(\mathcal{L}_i)$*, where* $S(\mathcal{L}_i)$ *denotes the symmetric group on the finite set* $\mathcal{L}_i$ *and* Π *denotes direct product. Thus, we have a natural representation of* K *as a permutation group on an infinite set, where every orbit is finite. The last part of the argument in Proposition 3.5 shows that this representation is faithful, since any element stabilizing every element of* $\mathcal{L}$ *must be the identity.*

3.2. **PD Groups.** Boundaries ∂G of PD(n) hyperbolic groups G are locally connected homological manifolds (over the integers) with the homology of a sphere of dimension $(n-1)$ by Theorem 1.13. If $QI(G)$ denotes the group of (boundary values of) quasi-isometries of G acting on ∂G, then Theorem 1.14 implies the following.

Proposition 3.9. $QI(G)$ *cannot have arbitrarily small torsion elements.*

We combine Corollary 2.7 with Proposition 3.9 below.

Proposition 3.10. *Let G be a Poincare duality hyperbolic group and and H an infinite quasiconvex subgroup of infinite index in G. Let K be a compact group of (boundary values of) pattern-preserving quasi-isometries of G. Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Then K must be finite.*

Proof: By Proposition 3.9, K cannot have arbitrarily small torsion elements. Hence, if K is infinite, it must have an element g of infinite order. Let $C(g)$ be the (closed) monothetic subgroup generated by g. Since K is totally disconnected by Proposition 3.7 so is $C(g)$ and hence $C(g)$ cannot have arbitrarily small torsion elements. By Theorem 3.4 $C(g)$ must contain a copy of the p-adic integers. But K cannot contain a copy of the p-adic integers by Corollary 2.7, a contradiction. Hence K is finite. $\Box$

We come now to the main Theorem of this section. Since G acts on its Cayley graph Γ by isometries, we are interested in uniform patter-preserving groups containing G.

Theorem 3.11. *Let G be a hyperbolic Poincare duality group and H an infinite quasiconvex subgroup of infinite index in G. Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Let Q be a group of pattern-preserving uniform quasi-isometries containing G. Then G is of finite index in Q. In particular, $Q \subset Comm(G)$, where $Comm(G)$ denotes the abstract commensurator of G.*

Proof: Let L be the limit set of H and $\mathcal{L}$ be the collection of translates of L under G. By Corollary 1.10, we can choose a finite collection $L_1 \cdots L_n$ of elements of $\mathcal{L}$ such that $Q_0 = \cap_{i=1 \cdots n} Stab(L_i)$ is compact, where $Stab(L_i)$ denotes the stabilizer of L_i in Q. Then Q_0 is finite by Proposition 3.10. As in the proof of Corollary 1.11, we can choose a neighborhood U of the identity in Q such that $U \subset Q_0$. Hence U_0 is finite and Q is discrete.

Let $Gq_1, \cdots Gq_n, \cdots$ be distinct cosets. Since G acts transitively on (the vertex set of) Γ, we can choose representatives $g_1 q_1, \cdots g_n q_n, \cdots$ such that $g_i q_i(1) = 1$ for all i. Since (the vertex set of) Γ is locally finite, the sequence $g_1 q_1, \cdots g_n q_n, \cdots$ must have a convergent subsequence in Q. Since Q is discrete, it follows that such a sequence must be finite. Hence G is of finite index in Q.

Since each element q of Q commensurates G to qGq^{-1} of finite index in Q, therefore $G \cap qGq^{-1}$ is of finite index in Q. Hence $G \cap qGq^{-1}$ is of finite index in G and qGq^{-1}, i.e. $q \in Comm(G)$, where $Comm(G)$ denotes the abstract commensurator of G. $\Box$

In fact the proof of Theorem 3.11 gives:

Corollary 3.12. *Let G be a hyperbolic Poincare duality group and H an infinite quasiconvex subgroup of infinite index in G. Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Let Q be a locally compact group of pattern-preserving quasi-isometries containing G. Then G is of finite index in Q. In particular, $Q \subset Comm(G)$, where $Comm(G)$ denotes the abstract commensurator of G.*

In the present context, Scholium 1.2 translates to the following precise statement as a consequence of Corollary 3.12.

Corollary 3.13. *Let ϕ be a pattern-preserving quasi-isometry between pairs (G_1, H_1) and (G_2, H_2) of hyperbolic PD groups and infinite quasiconvex subgroups of infinite index. Suppose d is a visual metric on ∂G_1 with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G_1, d)$. Further, suppose that G_1 and $\partial\phi^{-1} \circ G_2 \circ \partial\phi$ embed in some locally compact subgroup Q of $Homeo(\partial G_1)$ with the uniform topology. Then G_1 and G_2 are commensurable.*

4. Filling Codimension One Subgroups and Pattern Rigidity

4.1. **Codimension One Subgroups and Pseudometrics.** Let G be a one-ended Gromov-hyperbolic group with Cayley graph Γ. Let H be a quasiconvex subgroup. We say that H is codimension one if the limit set L_H of H disconnects ∂G. This is equivalent to saying that the join $J(L_H) = J$ disconnects Γ coarsely, i.e. if D be the quasiconvexity constant of J, then $\Gamma \setminus N_D(J)$ has more than one unbounded component, where $N_D(J)$ denotes the D-neighborhood of J. We say further that H is **filling** if for any two $x, y \in \partial G$, there exists a translate gL_H of L_H by an element g of G such that x, y lie in distinct components of $\partial G \setminus gL_H$. Let D_1 be such that any path joining points in distinct unbounded components of $\Gamma \setminus N_D(J)$ passes within D_1 of J. We say that $x, y \in \Gamma$ are separated by some translate gJ of J if x, y lie in distinct unbounded components of $\Gamma \setminus gN_{D+D_1}(J)$. Equivalently, we shall say that the geodesic $[x, y]$ is separated by some translate gJ of J.

Lemma 4.1. *Let H be a codimension one, filling, quasiconvex subgroup of a one-ended hyperbolic group G. Let Γ be a Cayley graph of G. There exists $C \geq 0$ such that any geodesic σ in Γ of length greater than C is separated by a translate of J.*

Proof: Suppose not. Then there exists a sequence of geodesic segments $\sigma_i = [a_i, b_i]$ which are not separated by any translate of J such that $d(a_i, b_i) \to \infty$. By equivariance, we may assume that σ_i is centered at the origin, i.e. $d(a_i, 1) \to \infty$ and $d(1, b_i) \to \infty$ and $1 \in [a_i, b_i]$. Let $a_i \to a_\infty \in \partial G$ and $b_i \to b_\infty \in \partial G$. Then a_∞ and b_∞ cannot lie in distinct components of $\partial G \setminus gL_H$ for any $g \in G$, for if they did then there exists $g \in G$ such that a_∞ and b_∞ lie in distinct components of $\partial G \setminus gL_H$ and hence for all i sufficiently large, a_i, b_i would lie in distinct unbounded components of $\Gamma \setminus gJ$. But if a_∞ and b_∞ cannot be separated, then H cannot be filling, contradicting the hypothesis. $\Box$

Lemma 4.2. *Let G, H, Γ, J be as above. Let $[a, b] \subset \Gamma$ be a geodesic and $c \in [a, b]$ such that $d(a, c) \geq 2D$, $d(b, c) \geq 2D$, where D is the quasiconvexity constant of J. Suppose gJ separates a, c. Then gJ separates a, b.*

Proof: Suppose not. Then a, b lie in the same unbounded component of $\Gamma \setminus gN_D(J)$, whereas c lies in a different unbounded component of $\Gamma \setminus gN_D(J)$. Hence there is a subsegment $[ecf]$ of $[a, b]$ such that $e, f \in N_{D_1}J$, but $c \notin N_{D+D_1}J$, contradicting the quasiconvexity constant for J. $\Box$

Lemma 4.3. [GMRS97] *Let G be a hyperbolic group and H a quasiconvex subgroup, with limit set L. Let J denote the join of the limit set. There exists $N \in \mathbb{N}$ such that there exist at most N distinct translates of J intersecting the 2-neighborhood $B_2(g)$ nontrivially for any $g \in G$.*

Lemma 4.4. *Define a new pseudometric ρ on Γ by declaring $\rho(a, b)$ to be the number of copies of joins $J \in \mathcal{J}$ separating a, b. Then (Γ, ρ) is quasi-isometric to (Γ, d)*

Proof: By Lemma 4.3, it follows that there exists $N \in \mathbb{N}$ such that $d(a,b) \leq C_0$ implies $\rho(a,b) \leq NC_0$. From Lemma 4.1, it follows that there exists $C_2 \geq 0$, such that $d(a,b) \geq C_2$ implies $\rho(a,b)) \geq 1$. Now from Lemma 4.2, it follows that for $n \in \mathbb{N}$, $d(a,b) \geq nC_2$ implies $\rho(a,b)) \geq n$. Hence the Lemma$\Box$

A purely topological version of Lemma 4.4 may be obtained as follows. Let $\partial^3 G$ denote the collection of distinct unordered triples of points on ∂G. Then it is well known [Gro85] [Bow98b] that G acts cocompactly on $\partial^3 G$ with metrizable quotient. Let d be any equivariant metric on $\partial^3 G$. Then by the Milnor-Svarc Lemma [GdlH90], $(\partial^3 G, d)$ is quasi-isometric to Γ. We say that a translate $gL \in \mathcal{L}$ separates closed subsets $A, B \subset \partial G$ if A, B lie in distinct components of $\partial G \setminus gL$. Define a pseudometric ρ on $\partial^3 G$ by defining $\rho(\{a_1,a_2,a_3\},\{b_1,b_2,b_3\})$ to be the number of copies of limit sets $gL \in \mathcal{L}$ separating $\{a_1,a_2,a_3\},\{b_1,b_2,b_3\}$. Then $(\partial^3 G, \rho)$ is quasi-isometric to (Γ, ρ), and hence to (Γ, d) and $(\partial^3 G, d)$. We state this as follows.

Corollary 4.5. *$(\partial^3 G, \rho)$, (Γ, ρ), (Γ, d) and $(\partial^3 G, d)$ are quasi-isometric to each other.*

4.2. **Pattern Rigidity.** We prove the following:

Proposition 4.6. *Let G be a one-ended hyperbolic group and H a codimension one, filling, quasiconvex subgroup. Then any pattern-preserving group Q of quasi-isometries is uniform.*

Proof: Assume without loss of generality that $G \subset Q$. Since any pattern-preserving homeomorphism of ∂G preserves (Γ, ρ) on the nose, it follows from Lemma 4.4 that Q is uniform. $\Box$

Combining Proposition 4.6 with Theorem 3.11 we get

Theorem 4.7. *Let G be a PD hyperbolic group and H a codimension one, filling, quasiconvex subgroup. Let Q be any pattern-preserving group of quasi-isometries containing G. Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Then the index of G in Q is finite.*

In fact more is true. Combining Proposition 4.6 with Corollary 4.5, we get

Proposition 4.8. *Let G be a one-ended hyperbolic group and H a codimension one, filling, quasiconvex subgroup with limit set L. Let $\mathcal{L}$ be the collection of translates of L under G. Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Then any pattern-preserving group Q of homeomorphisms of ∂G preserving $\mathcal{L}$ can be realized as the boundary values of uniform quasi-isometries.*

Combining Proposition 4.8 with Theorem 3.11 we finally get

Theorem 4.9. Topological Pattern Rigidity *Let G be a PD hyperbolic group and H a codimension one, filling, quasiconvex subgroup with limit set L. Let $\mathcal{L}$ be the collection of translates of L under G. Suppose d is a visual metric on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$. Let Q be any pattern-preserving group of homeomorphisms of ∂G preserving $\mathcal{L}$ and containing G. Then the index of G in Q is finite.*

Theorem 4.9 is a generalization of a Theorem of Casson-Bleiler [CB88] and Kapovich-Kleiner [KK00] to all dimensions. Casson-Bleiler [CB88] and Kapovich-Kleiner [KK00] proved Theorem 4.9 for G the fundamental group of a surface and H an infinite cyclic subgroup corresponding to a filling curve.

5. Relative Hyperbolicity

Let X be a geodesic metric space with a collection $\mathcal{H}$ of uniformly separated subsets $\{H_i\}$. The hyperbolic cone cH_i is the product of H_i and the non-negative reals $H_i \times \mathbb{R}_+$, equipped with the metric of the type $2^{-t}ds^2 + dt^2$. More precisely, $H_i \times \{n\}$ is given the path metric of H_i scaled by 2^{-n}. The $\mathbb{R}_+$ direction is given the standard Euclidean metric. Let $\mathcal{G}(X, \mathcal{H})$ denote X with hyperbolic cones cH_i glued to it along H_i's. X^h will be referred to as the *hyperbolically coned off* X. Gromov's definition of **(strong) relative hyperbolicity** is as follows.

Definition 5.1. *X is said to be strongly hyperbolic relative to the collection $\mathcal{H}$ in the sense of Gromov if the hyperbolically coned off space $\mathcal{G}(X, \mathcal{H})$ is a hyperbolic metric space.*

A closely related definition was given by Farb [Far98] and its equivalence to Gromov's definition established by Bowditch [Bow97].

5.1. **Symmetric Patterns of Horoballs.** The boundary of $\mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})$ will be called the relative hyperbolic boundary of $(G, \mathcal{H})$ and denoted as $\partial(G, \mathcal{H})$. Also, we shall denote the collection of hyperbolic cones $cH \subset \mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})$ as $c\mathcal{H}$ and refer to indivdual elements of $c\mathcal{H}$ as *horoballs*. The collection $c\mathcal{H}$ will be called a symmetric pattern (of horoballs). It is a fact that elements of $c\mathcal{H}$ are uniformly quasiconvex [BDM09] and that for any two distinct $cH_1, cH_2 \in c\mathcal{H}$, there is a coarsely well-defined 'centroid', i.e. the shortest geodesic joining $cH_1, cH_2 \in c\mathcal{H}$ is coarsely well-defined (any two such lie in a uniformly bounded neighborhood of each other [Far98]) and hence its mid-point (the centroid of cH_1, cH_2) is coarsely well-defined.

We now recast the relevant definitions and propositions of Sections 1 and 3 in the context of relative hyperbolicity. Let Γ, Γ_H, $\Gamma_{\mathcal{H}}$ denote respectively the Cayley graph of G, some translate of the Cayley (sub)graph of H and the collection of translates of Γ_H for $H \in \mathcal{H}$.

Definition 5.2. *The group $PP(G, H)$ of pattern-preserving maps for a (strongly) relatively hyperbolic pair $(G, \mathcal{H})$ as above is defined as the group of homeomorphisms of $\partial(G, \mathcal{H})$ preserving the base-points of $c\mathcal{H}$. The group $PPQI(G, H)$ of pattern-preserving quasi-isometries for a (strongly) relatively hyperbolic pair $(G, \mathcal{H})$ as above is defined as the subgroup of $PP(G, H)$ consisting of homeomorphisms h of ∂G such that $h = \partial\phi$ for some quasi-isometry $\phi : \mathcal{G}(\Gamma, \Gamma_{\mathcal{H}}) \to \mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})$ that permutes the collection of horoballs $c\mathcal{H}$.*

The following Theorem was proven in [Mj08] using the notion of *mutual coboundedness.*

Theorem 5.3. [Mj08] *Let G be a finitely generated group (strongly) hyperbolic relative to a finite collection $\mathcal{H}$ of finitely generated subgroups H. Let Γ, Γ_H, $\Gamma_{\mathcal{H}}$ denote respectively the Cayley graph of G, some translate of the Cayley (sub)graph of H and the collection of translates of Γ_H for $H \in \mathcal{H}$. There exist two elements cH_1, cH_2 of $c\mathcal{H}$ such that the following holds.*

For any K, ϵ, there exists a C such that if $\phi : \mathcal{G}(\Gamma, \Gamma_{\mathcal{H}}) \to \mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})$ is a pattern-preserving (K, ϵ)-quasi-isometry with $\partial\phi(\partial cH_i) = \partial cH_i$ for $i = 1, 2$, then $d(\phi(1), 1) \leq C$.

Let $\overline{\mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})}$ denote the Gromov compactification of $\mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})$ and $\overline{c\mathcal{H}}$ denote the collection of compactified horoballs, i.e. $\underline{\text{horoballs}}$ with basepoints adjoined. Let d_c denote a metric giving the topology on $\overline{\mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})}$. In this context Proposition 1.9 translates to the following (see [Mj08] for instance).

Proposition 5.4. *The collection $\overline{c\mathcal{H}}$ is discrete in the Hausdorff topology on the space of closed subsets of $\overline{\mathcal{G}(\Gamma, \Gamma_{\mathcal{H}})}$, i.e. for all $cH \in c\mathcal{H}$, there exists $\epsilon > 0$ such that $N_\epsilon(cH) \cap c\mathcal{H} = cH$, where $N_\epsilon(cH)$ denotes an ϵ neighborhood of cH in the Hausdorff metric arising from d_c.*

Let $Q \subset PPQI(G, H)$ be a group of quasi-isometries preserving a symmetric pattern of horoballs. Using Theorem 5.3 and Proposition 5.4, we have as in Section 3 (cf Proposition 3.5, Corollary 3.7, Proposition 3.9, Proposition 3.10)

Proposition 5.5. *Q has no non-trivial topologically infinitely divisible elements. More generally, Q does not contain any non-trivial infinitely divisible subgroups. Hence Q is totally disconnected. Suppose further $\partial(G, \mathcal{H})$ is a cohomology manifold and d is a visual metric on $\partial(G, \mathcal{H})$ with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial(G, \mathcal{H}), d)$. If Q is compact, then Q is finite.*

Now, let $Q_u \subset PPQI(G, H)$ be a group of *uniform* quasi-isometries preserving a symmetric pattern of horoballs. Then Q_u is locally compact (by Lemma 1.4) and contains a compact open subgroup K. Suppose further that $\partial(G, \mathcal{H})$ is a cohomology manifold (e.g. if G is the fundamental group of a complete negatively curved manifold of finite volume and H is the cusp group) and d is a visual metric on $\partial(G, \mathcal{H})$ with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial(G, \mathcal{H}), d)$. Then K is finite. Hence Q_u is discrete. Thus as in Theorem 3.11 we get the following.

Theorem 5.6. *Let G be a finitely generated group (strongly) hyperbolic relative to a finite collection $\mathcal{H}$ of finitely generated subgroups H. Suppose further that $\partial(G, \mathcal{H})$ is a cohomology manifold and d is a visual metric on $\partial(G, \mathcal{H})$ with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial(G, \mathcal{H}), d)$. Let Q be a group of uniform quasi-isometries containing G preserving a symmetric pattern of horoballs. Then G is of finite index in Q. In particular, $Q \subset Comm(G)$, where $Comm(G)$ denotes the abstract commensurator of G.*

5.2. **Weak QI rigidity for Relatively Hyperbolic Groups.** We shall be using the following Theorem of Behrstock-Drutu-Mosher (which follows from the proof of Theorem 4.8 of [BDM09]) in conjunction with Theorem 5.6 to get somewhat stronger results.

Theorem 5.7. (Behrstock-Drutu-Mosher [BDM09]**)** *Let G be a finitely generated group (strongly) hyperbolic relative to a finite collection $\mathcal{H}$ of finitely generated subgroups H for which each $H \in \mathcal{H}$ is not (strongly) relatively hyperbolic. Let Γ, Γ_H, $\Gamma_{\mathcal{H}}$ denote respectively the Cayley graph of G, some translate of the Cayley (sub)graph of H and the collection of translates of Γ_H for $H \in \mathcal{H}$. Then for every $L \geq 1$ and $C \geq 0$ there exists $R = R(L, C, G, \mathcal{H})$ such that the following holds.*

For any (L,C)- (self) quasi-isometry q of G, the image $q(\Gamma_H)$ is at a bounded Hausdorff distance R of some $\Gamma_H \in \Gamma_{\mathcal{H}}$.

Let q be a (self) quasi-isometry of Γ. Since elements of the collection $\Gamma_{\mathcal{H}}$ are mapped bijectively to bounded neighborhoods of elements of collection $\Gamma_{\mathcal{H}}$, q extends to a (self) quasi-isometry q^h of $\mathcal{G}(\Gamma,\Gamma_{\mathcal{H}})$ where the elements of $c\mathcal{H}$ are bijectively mapped to uniformly bounded neighborhoods of elements of $c\mathcal{H}$. Each element of $c\mathcal{H}$ has a unique limit point in $\partial(G,\mathcal{H})$ which we shall call its base-point. Let ∂q denote the induced map of $\partial(G,\mathcal{H})$ and $\partial\mathcal{H}$ denote the collection of base-points of $c\mathcal{H}$ in $\partial(G,\mathcal{H})$. Thus a simple consequence of Theorem 5.7 is the following.

Corollary 5.8. *Let G be a finitely generated group (strongly) hyperbolic relative to a finite collection $\mathcal{H}$ of finitely generated subgroups H for which each $H \in \mathcal{H}$ is not (strongly) relatively hyperbolic. Let Γ, Γ_H, $\Gamma_{\mathcal{H}}$ denote respectively the Cayley graph of G, some translate of the Cayley (sub)graph of H and the collection of translates of Γ_H for $H \in \mathcal{H}$. Then for every $L \geq 1$ and $C \geq 0$ there exist $L_1 \geq 1$, $C_1 \geq 0$ and $R = R(L,C,G,\mathcal{H})$ such that the following holds.*
For any (L,C)- (self) quasi-isometry q of G, there is an (L_1,C_1) (self) quasi-isometry q^h of $\mathcal{G}(\Gamma,\Gamma_{\mathcal{H}})$ such that the image $q^h(\Gamma_H)$ is at a bounded Hausdorff distance R of some $\Gamma_H \in \Gamma_{\mathcal{H}}$. Hence q induces a homeomorphism ∂q of $\partial(G,\mathcal{H})$ preserving the base-points of $c\mathcal{H}$.

Combining Corollary 5.8 with Theorem 5.6 we get the following.

Theorem 5.9. *Let G be a finitely generated group (strongly) hyperbolic relative to a finite collection $\mathcal{H}$ of finitely generated subgroups H such that*
1) each $H \in \mathcal{H}$ is not (strongly) relatively hyperbolic
2) $\partial(G,\mathcal{H})$ is a cohomology manifold and d is a visual metric on $\partial(G,\mathcal{H})$ with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial(G,\mathcal{H}),d)$.
Let Γ be a Cayley graph of G with respect to a finite generating set. Let Q be a group of uniform quasi-isometries of Γ containing G. Then G is of finite index in Q. In particular, $Q \subset Comm(G)$, where $Comm(G)$ denotes the abstract commensurator of G.

6. Consequences

6.1. **Quasi-isometric Rigidity.** Let $\mathcal{G}$ be a graph of groups with Bass-Serre tree of spaces $X \to T$. Let $G = \pi_1\mathcal{G}$. Let $\mathcal{VE}(T)$ be the set of vertices and edges of T . The metric on T induces a metric on $\mathcal{VE}(T)$, via a natural injection $\mathcal{VE}(T) \to T$ which takes each vertex to itself and each edge to its midpoint. Let d_H denote Hausdorff distance.

Combining Theorems 1.5, 1.6 of Mosher-Sageev-Whyte [MSW04] with the Pattern Rigidity theorem 4.7 we have the following QI-rigidity Theorem along the lines of Theorem 7.1 of [MSW04].

(We refer the reader to [MSW04] specifically for the following notions:
1) Depth zero raft.
2) Crossing graph condition.
3) Coarse finite type and coarse dimension.
4) Finite depth.)

Theorem 6.1. *Let $\mathcal{G}$ be a finite, irreducible graph of groups such that for the associated Bass-Serre tree T of spaces the vertex groups are $PD(n)$ hyperbolic groups for some fixed n and edge groups are filling codimension one in the adjacent vertex groups and that $\mathcal{G}$ is of finite depth. Further suppose that each vertex group G admits a visual metric d on ∂G with $dim_H < dim_t + 2$, where dim_H is the Hausdorff dimension and dim_t is the topological dimension of $(\partial G, d)$.*
If H is a finitely generated group quasi-isometric to $G = \pi_1(\mathcal{G})$ then H splits as a graph $\mathcal{G}'$ of groups whose depth zero vertex groups are commensurable to those of $\mathcal{G}$ and whose edge groups and positive depth vertex groups are respectively quasi-isometric to groups of type (a), (b).

Proof: By the restrictions on the vertex and edge groups, it automatically follows that all vertex and edge groups are PD groups of coarse finite type. Since the edge groups are filling, the crossing graph condition of Theorems 1.5, 1.6 of [MSW04] is satisfied. $\mathcal{G}$ is automatically finite depth, because an infinite index subgroup of a $PD(n)$ groups has coarse dimension at most $n-1$.

Then by Theorems 1.5 and 1.6 of [MSW04], H splits as a graph of groups $\mathcal{G}'$ with depth zero vertex spaces quasi-isometric to the vertex groups of $\mathcal{G}$ and edge groups quasi-isometric to the edge groups of $\mathcal{G}$. Further, the quasi-isometry respects the vertex and edge spaces of this splitting, and thus the quasi-actions of the vertex groups on the vertex spaces of $\mathcal{G}$ preserve the patterns of edge spaces.

By Theorem 4.7 the depth zero vertex groups in $\mathcal{G}'$ are commensurable to the corresponding groups in $\mathcal{G}$. $\Box$

6.2. **The Permutation Topology.** In this paper we have ruled out three kinds of elements from the group $PPQI(G, H)$ of pattern-preserving quasi-isometries under appropriate hypotheses on G:
a) Elements that admit arbitrarily small roots (topologically divisible elements)
b) Arbitrarily small torsion elements (essentially Theorem 1.14)
c) Elements with arbitrarily large powers close to the identity (no copies of the p-adics)

In a sense (a) and (c) are phenomena that are opposite to each other. In hindsight, the previous works on pattern rigidity [Sch97] [BM08] [Bis09] exploited (a) in the context of an ambient Lie group which automatically rules out (b) and (c).

We have used the fact that the group is pattern-preserving in a rather weak sense, only to conclude that the group we are interested in is totally disconnected. In fact Theorem 3.11 generalizes readily to show that a locally compact totally disconnected group of quasi-isometries containing G must be a finite extension of G under appropriate hypotheses on G. The crucial hypothesis is local compactness on $PPQI(G, H)$ which can be removed under hypotheses on H as in Theorem 4.7. We would like to remove the hypothesis of local compactness in more general situations.

Remark 3.8 gives a reasonably explicit structure of K for K a compact group of pattern-preserving quasi-isometries. K acts on the discrete set $\mathcal{L}$ of patterns with finite orbits $\mathcal{L}_1, \mathcal{L}_2, \cdots$ and hence $K \subset \Pi_i S(\mathcal{L}_i)$, where $S(\mathcal{L}_i)$ denotes the symmetric group on the finite set $\mathcal{L}_i$ and Π denotes direct product.

To establish Theorem 3.11 without the hypothesis of local compactness, two crucial problems remain:
Problem 1: A topological converse to Remark 3.8 which would say that a group

$K \subset \Pi_i S(\mathcal{L}_i)$ acting with finite orbits on $\mathcal{L}$ must be compact in the **uniform** topology on ∂G.

As an approach to this, we propose *an alternate topology* on $PPQI(G,H)$ and call it the **permutation topology**. Enumerate $\mathcal{L} = L_1, L_2, \cdots$. Since the representation of $PPQI(G,H)$ in the symmetric group of permutations $S(\mathcal{L})$ is faithful, we declare that a system of open neighborhoods of the origin in $PPQI(G,H)$ is given by the set U_N of elements of $PPQI(G,H)$ fixing $L_i, i = 1 \cdots N$. Now consider an element $\phi \in PPQI(G,H)$ acting with finite orbits on $\mathcal{L}$. Then the (closed) monothetic subgroup (ϕ) generated by ϕ is locally compact and by Corollary 2.7, under certain hypotheses, it cannot contain the p-adics. Hence it must have **arbitrarily small torsion elements**. We cannot apply Theorem 1.14 right away. To be able to apply Theorem 1.14, we need to show the following:
For ϵ as in Theorem 1.14, there exists N such that for all k, if ϕ^k stabilizes each $L_i, i = 1 \cdots N$, then each orbit of ϕ^k has diameter less than ϵ.
Thus a weak enough statement ensuring a comparison of the *permutation topology* with the **uniform topology** is necessary. The coarse barycentre construction of [Mj08] might be helpful here to construct quasi-isometries coarsely fixing large balls and providing a starting point for the problem.

Problem 2: A more important and difficult problem is to rule out elements of $PPQI(G,H)$ which fix finitely many elements of $\mathcal{L}$ (and hence coarsely fix the origin in G by Theorem 1.7) but act with at least one unbounded orbit on $\mathcal{L}$. We would have to show the following:
There exists N such that if $\phi \in PPQI(G,H)$ stabilizes each $L_i, i = 1 \cdots N$, then each orbit of ϕ is finite.

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